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The fully entangled fraction as an inclusive measure of entanglement applications

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Abstract

Characterizing entanglement in all but the simplest case of a two qubit pure state is a hard problem, even understanding the relevant experimental quantities that are related to entanglement is difficult. It may not be necessary, however, to quantify the entanglement of a state in order to quantify the quantum information processing significance of a state. It is known that the fully entangled fraction has a direct relationship to the fidelity of teleportation maximized under the actions of local unitary operations. In the case of two qubits we point out that the fully entangled fraction can also be related to the fidelities, maximized under the actions of local unitary operations, of other important quantum information tasks such as dense coding, entanglement swapping and quantum cryptography in such a way as to provide an inclusive measure of these entanglement applications. For two qubit systems the fully entangled fraction has a simple known closed-form expression and we establish lower and upper bounds of this quantity with the concurrence. This approach is readily extendable to more complicated systems. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A pure quantum state is entangled if it is impossible to factorize into a tensor product of states for the separate systems (e.g., the singlet state of two spin-1/2 particles, $(1/\sqrt{2})(|01\rangle - |10\rangle)$, is entangled). This property, originally introduced to sharpen discussions of foundational issues in quantum theory [1], has been

studied extensively with regard to nonlocal quantum correlations [2] indicated by the observed violation of Bell's inequality [3]. In the past decade, the focus of entanglement studies has shifted toward applications which use the nonclassical features of quantum systems to surpass classical limitations on communications and computation. Such applications are part of the emerging field of *quantum information* [4] and include quantum cryptography [5], dense coding [6], teleportation [7], entanglement swapping [8], and quantum computation [9].

Due to this recent interest in quantum entanglement applications, the characterization of entanglement in

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a mixed bipartite system has become an intensely studied problem. In general, mixed states are entangled if it is impossible to represent the density operator as an incoherent sum of factorizable pure states [10]. There are a number of measures of entanglement for a bipartite system. Three closely related measures are the entanglement of formation, the entanglement of distillation, and the concurrence. The entanglement of formation is defined as the least number of maximally entangled states required to asymptotically prepare a mixed state ρ with local operations and classical communications [11] and the entanglement of distillation is defined as the asymptotic yield of maximally entangled states that can be extracted from ρ with local operations and classical communications [11]. The concurrence [12] is monotonically related to the entanglement of formation, and therefore an equally valid measure of entanglement, but is the only measure described here that provides a closed expression for the simplest case of a two qubit bipartite system [13]. Relative entropy [14] measures entanglement by considering the ability to distinguish ρ from all separable states and negativity [15] quantifies the degree to which the eigenvalues of the partial transpose fail to satisfy the partial transpose separability condition [16]. To be sure all of these entanglement measures can be computed, like any physical quantity in quantum mechanics, from knowledge of the density matrix which can be found experimentally with tomography [17], but their relation to experimental consequences are indirect at best. For example, a two qubit mixed state described by an ensemble of partially entangled states can always be distilled, in a nonunique fashion, into a smaller ensemble of maximally entangled states which can in turn be used for useful quantum information processing [18].

Modern conventional wisdom holds that characterizing entanglement in all but the simplest of cases is a hard problem. Even understanding the relevant experimental quantities that are related to entanglement is difficult. It may not be necessary, however, to quantify the entanglement of a state in order to quantify the quantum information processing significance of a state. For example, Horodecki et al. [19] demonstrated that the maximum teleportation fidelity for a general two qubit system is given by

$$F_T^{\max} = \frac{1}{3}(1 + 2F), \quad (1)$$

where F is the fully entangled fraction [11] and is defined as the overlap between a mixed state $\hat{\rho}$ and a maximally entangled state $|\Phi\rangle$ maximized over all $|\Phi\rangle$:

$$F = \max_{|\Phi\rangle} \{ \langle \Phi | \hat{\rho} | \Phi \rangle \}. \quad (2)$$

Unlike entanglement, the fully entangled fraction does have a clear experimental interpretation as the optimal ability of a state to teleport and it is clear that the degree to which fully entangled fraction is greater than $1/2$ ($F_T > 2/3$) can be used to quantify the teleporting ability of a state over the best “classical teleportation” protocols. This suggests that it may be possible to define a *measure of entanglement applications* directly. Such a mathematical quantity may be just as useful as a true entanglement measure, but more practical from a theoretical standpoint. It is natural to wonder whether the fully entangled fraction is such a quantity, that is, can it be used to measure the general quantum information significance of a state. To answer this question for the case of two qubits, in Section 2, we examine the relationship between the fully entangled fraction and the fidelities of all two qubit applications which have been experimentally demonstrated to date: dense coding, teleportation, entanglement swapping, and quantum cryptography (Bell inequalities). We consider these applications with a general two qubit mixed state in place of the standard maximally entangled pure state and find that the fully entangled fraction does indeed quantify the quantum processing significance of dense coding, teleportation, entanglement swapping, and quantum cryptography (violating Bell’s inequality) in an inclusive sense. We are taking “inclusive” to mean that a nonzero value indicates that a state can perform at least one of these applications better than allowed “classically” and a zero value indicates that a state cannot perform any of these applications better than allowed “classically”. Similar approaches have examined continuous variable teleportation [20] and Bell inequality experiments [21]. The fully entangled fraction has a simple closed-form analytic expression, in the case of two qubits, which we rederive in Section 3 under the present context. In Section 4 we establish upper and lower bounds between the fully entangled fraction and the only measure of entanglement described above with a closed-form expression for a general two qubit state, the concurrence,

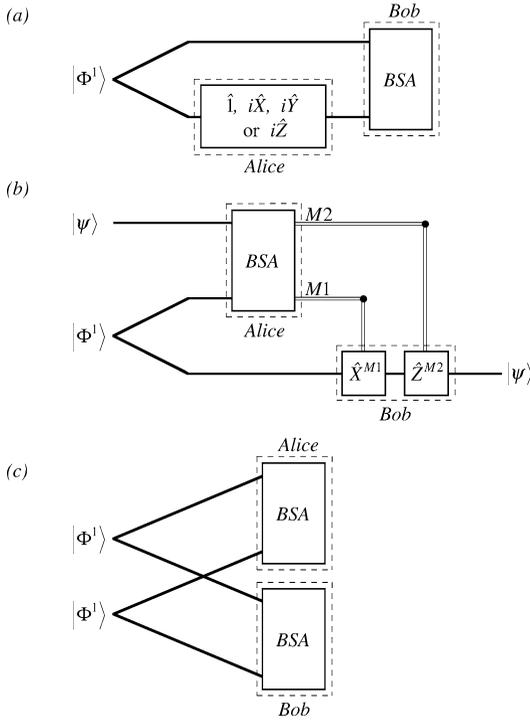


Fig. 1. Circuit diagram representation. (a) Dense coding: Alice applies one of four unitaries $\{\hat{I}, i\hat{X}, i\hat{Y}, i\hat{Z}\}$ to her qubit which Bob can read out with a Bell state analysis (BSA). (b) Teleportation: Alice teleports an unknown quantum state $|\psi\rangle$ by sending the result of a Bell state analysis $\{M1, M2\}$ to Bob who transforms his qubit into $|\psi\rangle$ conditioned on this information. (c) Entanglement swapping: Alice projects Bob's two particles into a maximally entangled state via a Bell state analysis on her two qubits. See Ref. [4] for a complete description of quantum circuit diagrams.

and in Section 5 we conclude and discuss generalizations of these ideas to more complicated systems.

2. The relation between two qubit applications and the maximally entangled fraction

2.1. Dense coding

The relationship between the fully entangled fraction and dense coding [6] (see Fig. 1(a)) is clearest. In this entanglement application Alice and Bob each receive one qubit of a maximally entangled state, $|\Phi^1\rangle \equiv \sqrt{1/2}(|00\rangle + |11\rangle)$, where the first entry denotes Bob's qubit and the second denotes Alice's qubit. Alice can encode 2 bits of information in four orthogonal states

by applying one of four local unitaries *solely* to her own qubit,

$$\begin{aligned} \hat{I} \otimes \hat{I} |\Phi^1\rangle &= |\Phi^1\rangle, \\ \hat{I} \otimes i\hat{X} |\Phi^1\rangle &= |\Phi^2\rangle \equiv i \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \\ \hat{I} \otimes i\hat{Y} |\Phi^1\rangle &= |\Phi^3\rangle \equiv -\frac{|01\rangle - |10\rangle}{\sqrt{2}}, \\ \hat{I} \otimes i\hat{Z} |\Phi^1\rangle &= |\Phi^4\rangle \equiv i \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \end{aligned} \quad (3)$$

where $\{\hat{X}, \hat{Y}, \hat{Z}\}$ are the Pauli operators [4] and $|\Phi^j\rangle$ ($j = 1, 2, 3, 4$) are the so-called “magic basis” states [11,12], an orthonormal set of maximally entangled states with a convenient phase convention. After receiving one qubit from Alice, Bob can read out two bits of information with a fidelity of one by performing a Bell state analysis on his two particles. To measure entanglement with this protocol the maximally entangled state $|\Phi^1\rangle$ is replaced with an arbitrary two qubit state $\hat{\rho}$ and the dense coding fidelity is defined as an average over the four possible outcomes:

$$\begin{aligned} F_{DC} = \frac{1}{4} & \left(\langle \Phi^1 | \hat{\rho} | \Phi^1 \rangle + \langle \Phi^2 | (\hat{I} \otimes i\hat{X}) \hat{\rho} (\hat{I} \otimes i\hat{X})^\dagger | \Phi^2 \rangle \right. \\ & + \langle \Phi^3 | (\hat{I} \otimes i\hat{Y}) \hat{\rho} (\hat{I} \otimes i\hat{Y})^\dagger | \Phi^3 \rangle \\ & \left. + \langle \Phi^4 | (\hat{I} \otimes i\hat{Z}) \hat{\rho} (\hat{I} \otimes i\hat{Z})^\dagger | \Phi^4 \rangle \right). \end{aligned} \quad (4)$$

Using the definition of the $|\Phi^j\rangle$ basis states, the dense coding fidelity reduces to the fidelity of $\hat{\rho}$ relative to a single maximally entangled state:

$$F_{DC} = \langle \Phi^1 | \hat{\rho} | \Phi^1 \rangle. \quad (5)$$

However, it is natural to expect that, experimentally, one should attempt to maximize the utility of the state by choosing the best possible local coordinate basis in which to carry out the experiment. Thus, the intrinsic capabilities of the state should take this into account. Mathematically, this is expressed by maximizing the quantity $\langle \Phi^1 | \hat{\rho} | \Phi^1 \rangle$ over all possible local unitary operators, viz.

$$F_{DC}^{\max} = \max_{\hat{U}_A, \hat{U}_B} \{ \langle \Phi^1 | (\hat{U}_A \otimes \hat{U}_B)^\dagger \hat{\rho} (\hat{U}_A \otimes \hat{U}_B) | \Phi^1 \rangle \}. \quad (6)$$

Because all maximally entangled states are related under local unitary operations, this is equivalent to maximizing $\langle \Phi | \hat{\rho} | \Phi \rangle$ over all maximally entangled

states $|\Phi\rangle$ which is just the fully entangled fraction:

$$F_{\text{DC}}^{\text{max}} = F. \quad (7)$$

It is clear that the maximum fidelity for dense coding ($F_{\text{DC}}^{\text{max}} = 1$) occurs when $\hat{\rho}$ is maximally entangled and the maximum fidelity for a separable state ($F_{\text{DC}}^{\text{max}} = 1/2$) occurs when $\hat{\rho}$ is pure.

2.2. Teleportation

For pedagogical reasons we next examine teleportation [7] (see Fig. 1(b)), whose relationship to the fully entangled fraction was first worked out by Horodecki et al. [19]. The goal of teleportation is to use a maximally entangled pair of qubits to transmit an arbitrary quantum state from one point to another with the communication of only two classical bits. Briefly, Alice has a qubit (particle 1) in an unknown quantum state $|\psi\rangle_1 = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$ ($0 < \theta < \pi$, $0 < \phi < 2\pi$), and Alice and Bob again share a maximally entangled state $|\Phi^1\rangle_{23}$ (particles 2 and 3, respectively). Alice performs a Bell state analysis on her two qubits (particles 1 and 2) measuring one of four possible outcomes $\{M1, M2\} \in \{0, 1\}$. Using two bits of classical information she informs Bob of the outcome and he applies the unitary transformation $\hat{Z}^{M2}\hat{X}^{M1}$, transforming his qubit into Alice's original quantum state with a fidelity of unity. Suppose that, instead of the maximally entangled state $|\Phi^1\rangle$, we attempt teleportation using an arbitrary two qubit state $\hat{\rho}$. Following Popescu [22], we define the teleportation fidelity as an ensemble average over all input states $|\psi\rangle$,

$$F_T = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi f(\theta, \phi) \sin(\theta) d\theta d\phi, \quad (8)$$

where $f(\theta, \phi) = \langle \psi | \hat{\rho}_{\text{out}} | \psi \rangle$. This quantity can be considered a measure of the usefulness of $\hat{\rho}$ for performing teleportation. It is easiest to compute by exchanging the measurements and control operations [23] so that the Bell basis transformation is followed by a controlled-NOT between particles 2 and 3 and a controlled-Z between particles 1 and 3 (see Ref. [4, Chapter 4], for a description of these quantum gates). The trace over Alice's system can be performed and the integral above can be computed to give

$$F_T = \frac{1}{3} (1 + 2\langle \Phi^1 | \hat{\rho} | \Phi^1 \rangle). \quad (9)$$

Again, maximizing this quantity over all over all possible local unitary operators gives

$$F_T^{\text{max}} = \frac{1}{3} (1 + 2F). \quad (10)$$

2.3. Entanglement swapping

The relationship between the fully entangled fraction and entanglement swapping [8] (see Fig. 1(c)) is similar to the case of dense coding. In this entanglement application, there are two pairs of maximally entangled states in a direct product state $|\Phi\rangle_{1234} = |\Phi^1\rangle_{12} \otimes |\Phi^1\rangle_{34}$ (where 1, 2, 3, and 4 label the particles, respectively). If Alice receives particles 1 and 3 and Bob receives particles 2 and 4, the state can be reexpressed in this basis as

$$|\Phi\rangle_{1234} = \frac{1}{2} \left(|\Phi^1\rangle_{13} \otimes |\Phi^1\rangle_{24} - |\Phi^2\rangle_{13} \otimes |\Phi^2\rangle_{24} + |\Phi^3\rangle_{13} \otimes |\Phi^3\rangle_{24} - |\Phi^4\rangle_{13} \otimes |\Phi^4\rangle_{24} \right). \quad (11)$$

A Bell measurement by Alice (Bob) will project Bob's (Alice's) particle into a maximally entangled state despite the fact that the two particles have never interacted in the past. Replacing either maximally entangled state with an arbitrary density matrix $\hat{\rho}$ and making use of symbolic manipulation software, the fidelity of entanglement swapping can be defined similarly to dense coding as an average over the fidelities of the four possible outcomes with a similar result:

$$F_{\text{ES}} = \langle \Phi^1 | \hat{\rho} | \Phi^1 \rangle. \quad (12)$$

Once again, maximizing this quantity over all over all possible local unitary operators gives

$$F_{\text{ES}}^{\text{max}} = F. \quad (13)$$

2.4. Quantum cryptography (Bell inequalities)

Last, we examine the relationship between the fully entangled fraction and Bell inequality experiments, which occur, for example, in the Ekert protocol for secure key distribution [5]. The standard Bell correlation function [24] is given by

$$B = \left| \text{Tr} \left\{ \hat{S}_1(\phi_1) \hat{S}_2(\phi_2) \hat{\rho} - \hat{S}_1(\phi_1) \hat{S}_2(\phi'_2) \hat{\rho} + \hat{S}_1(\phi'_1) \hat{S}_2(\phi_2) \hat{\rho} + \hat{S}_1(\phi'_1) \hat{S}_2(\phi'_2) \hat{\rho} \right\} \right|, \quad (14)$$

where $\hat{S}_j(\phi_j) = \cos(\phi_j)\hat{Z}_j + \sin(\phi_j)\hat{X}_j$. The Bell inequality is given by $B \leq 2$ and is violated when $B > 2$. The detectors are set to their optimal values $\{\phi_1 = 0, \phi'_1 = \pi/2, \phi_2 = \pi/4, \phi'_2 = 3\pi/4\}$ such that the violation is maximum for the maximally entangled state $|\Phi^1\rangle$. This state is then replaced with a general state $\hat{\rho}$ and the Bell correlation function as a function of $\hat{\rho}$ is given by

$$\begin{aligned} B(\hat{\rho}) &= \sqrt{2} \left| \text{Tr}\{(\hat{X} \otimes \hat{X} + \hat{Z} \otimes \hat{Z})\hat{\rho}\} \right| \\ &= \sqrt{2} \left| \sum_j \langle \Phi^j | (\hat{X} \otimes \hat{X} + \hat{Z} \otimes \hat{Z}) \hat{\rho} | \Phi^j \rangle \right| \\ &= 2\sqrt{2} \left| \langle \Phi^1 | \hat{\rho} | \Phi^1 \rangle - \langle \Phi^3 | \hat{\rho} | \Phi^3 \rangle \right|. \end{aligned} \tag{15}$$

It is clear that the normalized expression $B/2\sqrt{2}$, maximized over all local unitaries operating on the separate subsystems, will always be less than or equal to the fully entangled fraction. Therefore, a fully entangled fraction greater than $1/2$ is a sufficiency condition for violating Bell’s inequality. Munro et al. considered a similar situation by maximizing Bell correlations over all possible detector orientations $\{\phi_1, \phi'_1, \phi_2, \phi'_2\}$ [21]. We have verified numerically that this quantity is also always less than the fully entangled fraction by searching 500 000 random states weighted toward higher concurrences (we explain how this is done in Section 4). This result is not unexpected due to the fact that it is well known that there exist mixed states which can teleport arbitrary quantum states better than any classical protocol, yet fail to violate standard Bell inequalities [22].

It is interesting to note that in general, measures which maximize the overlap between a fiducial pure state and an input state with respect to a local basis, viz.

$$\begin{aligned} F(|\psi_f\rangle, \hat{\rho}) &= \max_{\hat{U}_A, \hat{U}_B} \{ \langle \psi_f | (\hat{U}_A \otimes \hat{U}_B)^\dagger \hat{\rho} (\hat{U}_A \otimes \hat{U}_B) | \psi_f \rangle \}, \end{aligned} \tag{16}$$

have the largest difference in fidelities between a maximally entangled state and a separable pure state,

$$\Delta = F(|\psi_f\rangle, |\Phi\rangle\langle\Phi|) - F(|\psi_f\rangle, |uv\rangle\langle uv|), \tag{17}$$

when $|\psi_f\rangle$ is maximally entangled. This can be seen by writing the fiducial state in a Schmidt decomposition $|\psi_f\rangle = (\hat{U}_A \otimes \hat{U}_B)(\cos(\theta/2)|00\rangle + \sin(\theta/2)|11\rangle)$

[25], maximizing each term in Eq. (17) separately over the local unitary operators, and then maximizing Δ with respect to θ to show that the maximum occurs for $\theta = \pi/2$. It is physically intuitive that this statement will also be true if this measure is generalized to fiducial mixed states. Although not a rigorous proof, this suggests that the fully entangled fraction is the “best” quantifier of entanglement applications in the sense of being the most inclusive.

3. A simple closed-form expression for the fully entangled fraction

In Section 2 we deduced that the fully entangled fraction can be physically interpreted as an inclusive measure of entanglement applications. That is, when F is greater than $1/2$ a mixed state can at least perform dense coding, teleportation, or entanglement swapping with a fidelity that is better than any separable state using classical protocols. It is clear that this quantity is invariant under local unitary operators, which can be viewed passively as a basis transformation, but not under local nonunitary operators (e.g., projective measurements and dissipation) [11,26]. These nonunitary operators can invoke irreversible changes in a state that are less useful for understanding the intrinsic properties of a quantum state. In light of this result we briefly reprise here the derivation of a closed-form expression, first derived by Bennett et al. [11], of the fully entangled fraction in the case of an arbitrary state of two qubits. We take as our starting point the fully entangled fraction as expressed by Eq. (6). This expression can be simplified by using a property of maximally entangled states, $\hat{U}_A \otimes \hat{U}_B |\Phi^1\rangle = \hat{1} \otimes \hat{U}_B \hat{Y} \hat{U}_A^\dagger \hat{Y} |\Phi^1\rangle$, and redefining the optimizing unitary $\hat{U} = \hat{U}_B \hat{Y} \hat{U}_A^\dagger \hat{Y}$, so that this expression involves only a single maximization over a local unitary operator:

$$F = \max_{\hat{U}} \{ \langle \Phi^1 | (\hat{1} \otimes \hat{U})^\dagger \hat{\rho} (\hat{1} \otimes \hat{U}) | \Phi^1 \rangle \}. \tag{18}$$

Expanding $|\Phi\rangle = (\hat{1} \otimes \hat{U})|\Phi^1\rangle$ in a Pauli basis and making use of the basis states defined in Eq. (1),

$$\begin{aligned} |\Phi\rangle &= \hat{1} \otimes (x_1 \hat{1} + ix_2 \hat{X} + ix_3 \hat{Y} + ix_4 \hat{Z}) |\Phi^1\rangle \\ &= \sum_{n=1}^4 x_n |\Phi^n\rangle, \end{aligned} \tag{19}$$

allows one to represent an arbitrary maximally entangled state by four real parameters x_n ($n = 1, 2, 3, 4$) that satisfy $g(x_1, x_2, x_3, x_4) \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. Inserting this expression into Eq. (18) gives

$$F(x_1, x_2, x_3, x_4) = \sum_{n,m=1}^4 M_{n,m} x_n x_m, \quad (20)$$

where $M_{n,m} = \langle \Phi^n | \hat{\rho} | \Phi^m \rangle$. The extrema condition is found by including the constraint with an undetermined Lagrange multiplier λ :

$$\frac{\partial}{\partial x_k} \{ F(x_1, x_2, x_3, x_4) + \lambda g(x_1, x_2, x_3, x_4) \} = 0. \quad (21)$$

This condition in conjunction with the hermiticity of $\hat{\rho}$ gives rise to an eigenvalue equation

$$\sum_{n=1}^4 \text{Re}\{M_{k,n}\} x_n = -\lambda x_k. \quad (22)$$

The eigenvectors $(x_1^j, x_2^j, x_3^j, x_4^j)$ ($j = 1, 2, 3, 4$) of this real, symmetric matrix are orthonormal since $g = 1$. Inserting the eigenvectors into Eq. (20) and making use of their ortho-normalization results in $F = \max\{\eta^j\}$, where $\eta^j = -\lambda^j$ are the corresponding eigenvalues of this matrix. It is convenient to renormalize this expression so that it is 1 for a maximally entangled state and 0 for a separable state,

$$E(\hat{\rho}) = 2 \left(\max\{\eta^j, 0\} - \frac{1}{2} \right), \quad (23)$$

where η^j are the eigenvalues of the matrix $M_{n,m} = \text{Re}\{\langle \Phi^n | \hat{\rho} | \Phi^m \rangle\}$, $|\Phi^j\rangle$ being the maximally entangled basis states defined in Eq. (1).

4. The relation between the fully entangled fraction and the concurrence

The fully entangled fraction, once measured, establishes lower and upper bounds for the concurrence. It has been proved that the fully entangled fraction is a lower bound for the entanglement of formation [11, 27] and therefore a lower bound for the concurrence which is monotonically related to the entanglement of formation. The states which form the lower bound are given by a convex sum of a maximally mixed state and

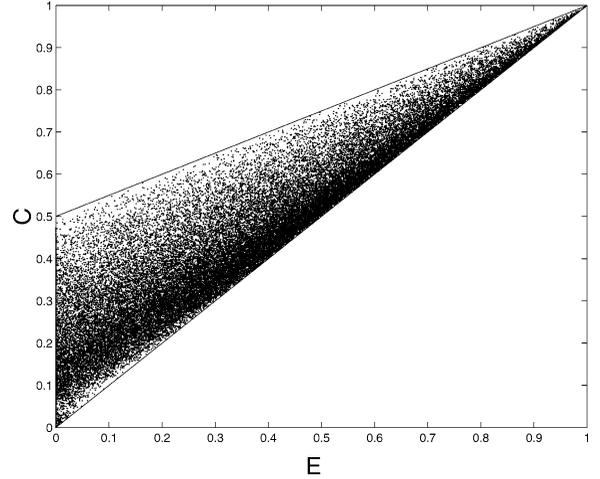


Fig. 2. Concurrence C vs. E for 100 000 random density matrices. In order to achieve a more uniform distribution and better demonstrate the upper and lower bounds, we plot a modified distribution consisting of a convex sum between Eqs. (25) and (26) with a parameter that varies between 0 and 0.5. The upper and lower bounds (solid lines) are given by $E = 2C - 1$ and $E = C$, respectively.

an arbitrary pure state,

$$\hat{\rho}_- = \epsilon \hat{1} + (1 - \epsilon) |\psi_{\text{pure}}\rangle \langle \psi_{\text{pure}}|, \quad (24)$$

where $0 < \epsilon < 1$. If the pure state is decomposed in a Schmidt basis, $|\psi_{\text{pure}}\rangle = (\hat{U}_A \otimes \hat{U}_B)(\cos(\theta/2)|00\rangle + \sin(\theta/2)|11\rangle)$ [25], the local unitaries will not contribute and we find that $E(\hat{\rho}_-) = C(\hat{\rho}_-) = (1 - \epsilon) \times \sin \theta - \epsilon/2$ (the maximum between this number and zero is implicit).

The upper bound for the concurrence is found numerically by doing a numerical search over one million random density matrices (see Fig. 2),

$$R = \frac{TT^\dagger}{\text{Tr}\{TT^\dagger\}}, \quad (25)$$

where T is a 4×4 matrix whose elements $T_{n,m} = t_r + it_i$ are determined by the random numbers t_r, t_i chosen uniformly on the interval $\{0, 1\}$. We find that the upper bound for the concurrence occurs for states that are a convex sum of a direct product state $|uv\rangle$ and a maximally entangled state $|\Phi\rangle$,

$$\hat{\rho}_+ = \zeta |uv\rangle \langle uv| + (1 - \zeta) |\Phi\rangle \langle \Phi|, \quad (26)$$

such that $\langle uv | \Phi \rangle = 0$ and $0 < \zeta < 1$. Taking $|uv\rangle = |01\rangle$ and $|\Phi\rangle = |\Phi^1\rangle$, we compute $E(\hat{\rho}_+) = 1 - 2\zeta$ and

$C(\hat{\rho}_+) = 1 - \zeta$, which implies $E(\hat{\rho}_+) = 2C(\hat{\rho}_+) - 1$. These bounds, taken together, imply that a nonzero E is a necessary condition for nonzero concurrence, but not a sufficient one. These results are consistent with similar entanglement of distillation results found by Bennett et al. [28]. We see that this operational measure determines the range of possible concurrence values C_{pos} for a mixed state:

$$E \leq C_{\text{pos}} \leq \frac{E + 1}{2}. \tag{27}$$

5. Conclusions

In conclusion, we have found that the fully entangled fraction can be used as an inclusive measure of entanglement applications in the case of two qubit states. That is, $F > 1/2$ guarantees that a mixed state can be used to achieve, on average, “classically impossible” results in either dense coding, teleportation, entanglement swapping, or quantum cryptography (Ekert protocol); all two qubit quantum information processing applications which have been experimentally demonstrated to date. This quantity has a simple closed-form expression for general two qubit states given by the largest eigenvalue of the real part of the density matrix expressed in a “magic” Bell basis. Although it appears that the fully entangled fraction is the “best” measure of entanglement applications in the sense of being the most inclusive, we leave this question open. It could be conceived that there are other two qubit applications or definitions of fidelity which have direct experimental consequences that include the fully entangled fraction as a subset. In which case it would define a new inclusive measure of these entanglement applications which sets the threshold for accomplishing classically inconceivable quantum information tasks. This quantity may be of more practical use than entanglement for characterizing the quantum informations processing ability of more complicated systems. For example, dense coding generalized to $d \times d$ systems allows Alice to use a maximally entangled state $|\Phi^1\rangle$ to encode $d^2/2$ bits in d^2 orthogonal states $|\Phi^i\rangle = (\hat{1} \otimes \hat{U}_i)|\Phi^1\rangle$ by applying d^2 local unitary operators \hat{U}_i (where $i = 1, \dots, d^2$). Replacing this maximally entangled state with a general density operator and defining the fidelity as in Section 2.1 as

an average of the d^2 results gives

$$F_{\text{DC}} = \frac{1}{d^2} \sum_{i=1}^{d^2} \langle \Phi^i | (\hat{1} \otimes \hat{U}_i) \hat{\rho} (\hat{1} \otimes \hat{U}_i)^\dagger | \Phi^i \rangle = \langle \Phi^1 | \hat{\rho} | \Phi^1 \rangle. \tag{28}$$

Maximizing this over all local unitaries (this is the same as maximizing over all maximally entangled states [29]) we see that the maximum fidelity of dense coding in this more general case is also given by the fully entangled fraction:

$$F_{\text{DC}}^{\text{max}} = F. \tag{29}$$

$F_{\text{DC}}^{\text{max}} = 1$ when $|\Phi^1\rangle$ is maximally entangled and $F_{\text{DC}}^{\text{max}} = 1/d$ when $|\Phi^1\rangle$ is pure and separable. Horodecki et al. [19] also found a similar result for the maximum teleportation fidelity:

$$F_T^{\text{max}} = \frac{Fd + 1}{d + 1}. \tag{30}$$

A general analytic expression for the fully entangled fraction for the general mixed case in this system is not known, however, there are known analytic results in the case of pure states [30]. It may also be possible to generalize the association of the fully entangled fraction with fidelities of quantum information tasks in multipartite systems.

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References

- [1] A. Einstein, B. Podolsky, N. Rosen, *Phys. Rev.* 47 (1935) 777; E. Schrödinger, *Proc. Cambridge Philos. Soc.* 31 (1935) 555.
- [2] J.S. Bell, *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy*, Cambridge University Press, New York, 1987; J.F. Clauser, A. Shimony, *Rep. Prog. Phys.* 41 (1978) 1881.
- [3] S.J. Freedman, J.F. Clauser, *Phys. Rev. Lett.* 28 (1972) 938; A. Aspect, P. Grangier, G. Roger, *Phys. Rev. Lett.* 49 (1982) 91;

- P.G. Kwiat et al., *Phys. Rev. A* 60 (1999) R773;
M.A. Rowe et al., *Nature* (London) 409 (2001) 791.
- [4] M.A. Nielsen, I.L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.
- [5] A.K. Ekert, *Phys. Rev. Lett.* 67 (1991) 661;
T. Jennewein et al., *Phys. Rev. Lett.* 84 (2000) 4729;
D.S. Naik et al., *Phys. Rev. Lett.* 84 (2000) 4733;
W. Tittel, J. Brendel, H. Zbinden, N. Gisin, *Phys. Rev. Lett.* 84 (2000) 4737.
- [6] C.H. Bennett, S.J. Wiesner, *Phys. Rev. Lett.* 69 (1992) 2881;
K. Mattle, H. Weinfurter, P.G. Kwiat, A. Zeilinger, *Phys. Rev. Lett.* 76 (1996) 4656.
- [7] C.H. Bennett et al., *Phys. Rev. Lett.* 70 (1993) 1895;
D. Bouwmeester et al., *Nature* (London) 390 (1997) 575;
D. Boschi et al., *Phys. Rev. Lett.* 80 (1998) 1121.
- [8] M. Zukowski, A. Zeilinger, M.A. Horne, A.K. Ekert, *Phys. Rev. Lett.* 71 (1993) 4287;
J.-W. Pan, D. Bouwmeester, H. Weinfurter, A. Zeilinger, *Phys. Rev. Lett.* 80 (1998) 3891;
T. Jennewein, G. Weihs, J.-W. Pan, A. Zeilinger, *quant-ph/0201134*.
- [9] D.P. DiVincenzo, *Science* 270 (1995) 255;
V. Vedral, M.B. Plenio, *Prog. Quantum Electron.* 22 (1998) 1.
- [10] R.F. Werner, *Phys. Rev. A* 40 (1989) 4277;
S.L. Braunstein et al., *Phys. Rev. Lett.* 83 (1999) 1054.
- [11] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, W.K. Wootters et al., *Phys. Rev. A* 54 (1996) 3824.
- [12] S. Hill, W.K. Wootters, *Phys. Rev. Lett.* 78 (1997) 5022.
- [13] W.K. Wootters, *Phys. Rev. Lett.* 80 (1998) 2245.
- [14] V. Vedral, M.B. Plenio, *Phys. Rev. A* 57 (1998) 1619.
- [15] G. Vidal, R.F. Werner, *Phys. Rev. A* 65 (2002) 032314.
- [16] A. Peres, *Phys. Rev. Lett.* 77 (1996) 1413;
M. Horodecki, P. Horodecki, R. Horodecki, *Phys. Lett. A* 223 (1996) 1.
- [17] A.G. White, D.F.V. James, P.H. Eberhard, P.G. Kwiat, *Phys. Rev. Lett.* 83 (1999) 3103;
D.F.V. James, P.G. Kwiat, W.J. Munro, A.G. White, *Phys. Rev. A* 64 (2001) 052312.
- [18] M. Horodecki, P. Horodecki, R. Horodecki, *Phys. Rev. Lett.* 78 (1997) 574.
- [19] M. Horodecki, P. Horodecki, R. Horodecki, *Phys. Rev. A* 60 (1999) 1888.
- [20] P.T. Cochrane, G.J. Milburn, W.J. Munro, *Phys. Rev. A* 62 (2000) 062307.
- [21] W.J. Munro, K. Nemoto, A.G. White, *J. Mod. Opt.* 48 (2001) 1239.
- [22] S. Popescu, *Phys. Rev. Lett.* 72 (1994) 797.
- [23] R.B. Griffiths, C.-S. Niu, *Phys. Rev. Lett.* 76 (1996) 3228.
- [24] J.F. Clauser, A. Shimony, *Rep. Prog. Phys.* 41 (1978) 1881.
- [25] A. Peres, *Quantum Theory: Concepts and Methods*, Kluwer, Dordrecht, 1993.
- [26] P. Badziag, M. Horodecki, P. Horodecki, R. Horodecki, *Phys. Rev. A* 62 (2000) 012311.
- [27] S. Ishizaka, *Phys. Rev. A* 63 (2001) 034301.
- [28] C.H. Bennett et al., *Phys. Rev. Lett.* 76 (1996) 722.
- [29] K.G.H. Vollbrecht, R.F. Werner, *J. Math. Phys.* 41 (2000) 6772.
- [30] G. Vidal, D. Jonathan, M.A. Nielsen, *Phys. Rev. A* 62 (2000) 012304.